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Judgments aggregation by a sequential majority procedure

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HIGHLIGHTS

- We introduce a *judgment aggregation procedure*, SMP, based on majority voting rule.
- The SMP is sequential and it is (roughly) characterized by *collective rationality* and restriction to majority rule whenever voting is required.
- The SMP can be axiomatized by the five properties: *Anonymity* (AN), *Restricted Agenda property* (RA), *Restricted Monotonicity* (RM), *Limited Neutrality* (LN), and *Independence of Past Deliberation* (IPD).
- The SMP satisfies the property of *Independence of Irrelevant Issues* (III) with respect to *relevance relation* naturally derived from the agenda.

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ABSTRACT

We consider a standard model of judgment aggregation as presented, for example, in Dietrich (2015). For this model we introduce a *sequential majority procedure* (SMP) which uses the majority rule as much as possible. The ordering of the issues is assumed to be exogenous. The definition of SMP is given in Section 2. In Section 4 we construct an intuitive *relevance relation* for our model, closely related to conditional entailment, for our model. While in Dietrich (2015), the relevance relation is given exogenously as part of the model, we insist that the relevance relation be derived from the agenda. We prove that SMP has the property of *independence of irrelevant issues* (III) with respect to (the transitive closure of) our relevance relation. As III is weaker than the property of proposition-wise independence (PI) we do not run into impossibility results as does List (2004) who incorporates PI in some parts of his analysis. We proceed to characterize SMP by anonymity, restricted monotonicity, limited neutrality, restricted agenda property, and independence of past deliberations (see Section 3 for the precise details). SMP inherits the first three axioms from the Majority Rule. The axiom of restricted agenda property guarantees sequentiality. The most important axiom, independence of past deliberations (IPD), says that the choice at time $(t+1)$ depends only on the choices in dates $1, \ldots, t$ and the judgments at $(t+1)$ (and not on the individual judgments in dates $1, \ldots, t$). Also, we use this occasion to point out that Roberts (1991) characterization of choice by plurality voting may be adapted to our model.

0. Introduction

In a judgment aggregation problem a group of two or more decision makers (agents, voters, judges etc.) have to make collective decisions on an *agenda* consisting of finitely many logically interconnected *issues*. An issue is a pair of a proposition with its negation. A *complete* judgment is a selection of one proposition from each issue in the agenda. A certain set of complete judgments is called the set of *rational judgments* (e.g. judgment with no logical contradictions or judgments that conform with the norms or tradition of the society). A judgment (not necessarily complete) is *consistent* if it is contained in a rational judgment. We assume that each agent has a rational judgment and the problem of judgment aggregation is how to combine the rational individual judgments into a collective rational judgment in a democratic society.

The theory started with the following example known as the *doctrinal Paradox*:

Example 1 (*The Doctrinal Paradox*)**.** Consider three judges deliberating on the following issues:

- p — The contract is legally valid (or: $\neg p$ — The contract is *not* legally valid).
- q — The defendant has broken the contract (or: $\neg q$ — The defendant has *not* broken the contract).
- g — The defendant is liable (or: $\neg g$ — The defendant is *not* liable).
- By law, $g \Leftrightarrow p \wedge q$ that is, the defendant is liable if and only if he/she has broken a valid contract.

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Assume that the judgments of the three judges are those given in the following table (where 1 indicates that the proposition is true and 0 indicates that it is false):

	Issues					
	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1

Note that the judgment of each judge is rational. Aggregation of the judgments by simple majority voting on each proposition yields the following:

	Issues					
	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1
	1	**0**	**1**	**0**	**0**	**1**

This aggregated judgment is **inconsistent** as p and q are accepted and yet $\neg g$ is also accepted. This 'paradox' is caused by using majority voting exclusively without taking into consideration *collective rationality*, which is part of our proposed aggregation procedure.

Inspired by this example, List and Petit (2002) proved a general impossibility theorem for judgment aggregation showing that certain intuitive assumptions may not be compatible with collective rationality. Their results have been followed by many papers with negative and positive results (see List (2012), Dietrich and List (2007a), Dokow and Holzman (2010) for a survey). We claim that this paradox may actually be solved rationally if the agenda is ordered: It is natural that $\{p, q\}$ is decided before g (the relative ordering of p and q does not matter). If this order is respected, the judges apply majority voting and impose consistency of their aggregated judgment (that is, $g \Leftrightarrow p \wedge q$) the result is (p, q, g). If they vote on g first the result is $(\neg g, p, \neg q)$ or $(\neg g, q, \neg p)$ depending on the order in which p and q are voted. We shall show, as this example indicates, that if the agenda is ordered we can always insist on collective rationality while being fully restricted to the majority voting rule. If the agenda is not ordered this cannot be done effectively as the resulting judgment may depend on the order in which the issues in the agenda are treated.

In this paper we investigate the family of judgment aggregation procedures that obey the following three conditions: (1) The only voting procedure that the agents may use is majority voting; (2) Majority voting is used in every step of the procedure that allows its use (the voters may form a collective judgment without voting, for example, by choosing a random dictator outside N; we exclude such possibilities); and (3) the procedure is 'collectively rational' that is, the outcome of the aggregation is a rational judgment. Let $N = \{1, \dots, n\}$, $n > 1$, be the set of decision makers and let $A = \{p_1, \neg p_1, \dots, p_k, \neg p_k\}$, $k > 1$, be the set of propositions that is their (temporally or otherwise) ordered agenda. Then the agents first choose between p_1 and $\neg p_1$ by majority. They cannot choose from a larger set because this may lead to cycles in certain cases (and of course, our method should work for every situation). Thus our procedure will be sequential. Suppose now that $q_1, \dots, q_h$ have been chosen, $h \geq 1$. Then we distinguish the following possibilities. If $p_1 \wedge, \dots \wedge p_h$ entails q for some q in the issue $I_{h+1} = \{p_{h+1}, \neg p_{h+1}\}$, then q is chosen in order to satisfy collective rationality. Otherwise, we choose from I_{h+1} by majority voting. Thus, ensuring collective rationality comes as a first consideration and only then, when possible, majority voting is applied. The detailed presentation of our procedure is given in Section 2. Of course, we insist on unrestricted domain of our procedure as is explained in Sections 1 and 2. We do not have uniqueness because, when the number of voters is even, arbitrary tie-breaking rules must be introduced.

In Section 1 we present the framework of judgment aggregation theory as it appears, for example, in Dietrich (2015). We also give some simple examples. The reader should read Example 3, the semantic model, carefully as we use it in later sections to present counterexamples to some of our conjectures. In Section 2 we define the sequential majority procedure (SMP) and give some further examples. Section 3 is devoted for an axiomatization of SMP. Our SMP inherits from the Majority Rule anonymity and modified versions of monotonicity (restricted monotonicity) and neutrality (limited neutrality). In addition it satisfies restricted agenda property that guarantees sequentiality. The last and most important axiom is *independence of past deliberations* (IPD). It implies that the social decision at date $t+1$ depends only on the choices at dates $1, \dots, t$ and the individual judgments at date $t+1$. Section 4 is devoted to the analysis of *relevance relations* as it has been developed by Dietrich. We focus on a relevance relation that is derived from conditional entailment, in particular, determined by the agenda. Actually, we have to take the transitive closure of this relation. As our relation is negation invariant, we are able to prove that SMP is independent of irrelevant issues with respect to (the transitive closure) of conditional entailment (as defined for ordered agendas). Our independence of irrelevant issues is much weaker than the classical proposition-wise independence. Finally, in Section 5, we mention a result of Roberts on plurality voting that is related to our model.

1. The model

There is a finite group of decision makers (or players) $N = \{1, 2, \dots, n\}$, $n \geq 2$. They are examining a set of propositions $X = \{p_1, \dots, p_k, \dots\}$ that may be finite or countably infinite. With each proposition $p \in X$ the negation of p, $\neg p$ is also in X. An agenda $A_k = \{p_1, \neg p_1, \dots, p_k, \neg p_k\}$ is a finite subset of X that contains with each proposition $q \in A_k$ its negation $\neg q$. An *issue* is a pair of propositions $I = \{p, \neg p\}$. Thus, the agenda is partitioned into a finite set of issues: $A_k = \{I_1, \dots, I_k\}$. A judgment J is a subset of A with the property that whenever $q \in J$, then $\neg q$ is not in J. A judgment J is *complete* if for each p not in J we have $\neg p \in J$. A certain nonempty set $\mathscr{J}$ of complete judgments is known to all voters as the set of *rational judgments*. A judgment J is *consistent* if it is contained in a rational judgment. A set of propositions $S \subset A$ *entails* a proposition $p \in A$, denoted by $S \models p$, if whenever S is contained in a rational judgment J, then $p \in J$. By this definition, the relation of entailment satisfies the following properties: for any propositions $p \in A$ and $q \in A$ and sets of propositions $S \subset A$ and $T \subset A$,

Monotonicity : If $S \models p$ and $T \supseteq S$ then $T \models p$.

Transitivity : If $S \models p$ and $S \cup \{p\} \models q$ then $S \models q$.

These two properties imply the following weaker version (due to the monotonicity) of transitivity:

Weak Transitivity : If $S \models p$ and $p \models q$ then $S \models q$.

To obtain significant results, the set of rational judgments must satisfy some minimal properties. To that end we make the following assumption (see Dietrich 2016).

Assumption.

– The set $\mathscr{J}$ of rational judgments has *no tautologies* that is, there is no proposition $p \in A$ such that $p \in J$ for all $J \in \mathscr{J}$.

– To avoid redundancy, we assume that there are *no equivalent propositions* that is, there is no $q \neq p$ such that $q \models p$ and $p \models q$. Equivalently, there are *no equivalent issues*.

This assumption also guarantees that the set $\mathscr{J}$ of rational judgments is "rich" enough in the sense that for each $p \in A$ there is $J \in \mathscr{J}$ such that $p \in J$.

Definition 1. A *judgment aggregation problem* (JAP) is a 5-tuple $g = (N, A_k, \neg, \wedge, \mathscr{J})$, where N is the set of voters (decision makers, judges, players, etc.), A_k is the agenda, $\neg$ and $\wedge$ are the symbols of negation and conjunction, respectively, and $\mathscr{J}$ is the set of *rational judgments*.

Definition 2. An *aggregation function* (AF) for a JAP is a function $F : \mathscr{J}^n \to \mathscr{J}$.

Example 2 (*Propositional Calculus*)**.** Let $\mathscr{L}$ be a propositional language on a given finite set of atoms $A = \{p_1, \ldots, p_k\}$, endowed with the following functions: for $p \in \mathscr{L}$, $\neg p$ (not p) (with $\neg p \neq p$ and $\neg\neg p = p$), $p_1 \wedge p_2$ (both p_1 and p_2), $p_1 \vee p_2$ (p_1 or p_2), and $p_1 \Rightarrow p_2$ (p_1 implies p_2). The set of *rational judgments* is the set of complete judgments with no logical contradictions.

Example 3 (*The Semantic Model (see, e.g., Dietrich, 2014), Section 2*)**.** This is a judgment aggregation problem $g = (N, A_k, \neg, \wedge, \mathscr{J})$ in which the propositions are subsets of a finite set $\Omega = \{a_1, a_2, \ldots, a_m\}$ and the negation of a proposition $p \subset \Omega$ is its complement w.r.t. Ω; $\neg p = \Omega \setminus p$. The *entailment* $\models$ is represented by set *inclusion* $\subset$, the *conjunction* $\wedge$ is represented by *intersection* $\cap$, and the *disjunction* $\vee$ is represented by set *union* $\cup$.

Example 4 (*Preference Aggregation*)**.** Given a set $S = \{a, b, \ldots\}$ of social alternatives, the preference aggregation problem is a JAP $g = (N, A_k, \neg, \wedge, \mathscr{J})$ in which the propositions are of the form $a \succ b$ (or $a \succeq b$). A judgment of a voter is his (complete and strict) preference order on the set of social alternatives, and consistency is imposed by the linearity of the (strict) preferences.

2. Sequential majority procedure (SMP)

Given a JAP $g = (N, A_k, \neg, \wedge, \mathscr{J})$ with an agenda consisting of k issues $A_k = \{I_1, \ldots, I_k\}$, when the issues are ordered (for example, temporally), we write a judgment as an ordered array $J = (q_1, \ldots, q_k)$ where $q_\ell \in I_\ell$; $\ell = 1, \ldots, k$, and we denote the following:

– $J_\ell = q_\ell$, the judgment for the ℓth issue I_ℓ.
– $J_{|\ell} = (q_1, \ldots, q_\ell)$, the judgment for the first ℓ issues $(I_1, \ldots, I_\ell)$ that is, $J_{|\ell} = \{J \cap (I_1 \cup, \ldots, \cup I_\ell)\}$.

For any profile $J^N \in \mathscr{J}^N$ and for $\ell = 1, \ldots, k$ we denote the following:

– $J^N_\ell = (J^1_\ell, \ldots, J^n_\ell)$, the profile of judgments for the issue I_ℓ.
– $\mathscr{J}^N_\ell$ is the set of profile of judgments for the issue I_ℓ.
– $\hat{\mathscr{J}}^N_\ell \subset \mathscr{J}^N_\ell$ is the set of profiles of judgments for the issue I_ℓ that result in *tie* between p_ℓ and $\neg p_\ell$ (which can happen only when n is even) that is,

$$\hat{\mathscr{J}}^N_\ell = \{J^N | \{i | J^i_\ell = p_\ell\}| = |\{i | J^i_\ell = \neg p_\ell\}|\}.$$

– $J^N_{|\ell} = (J^1_{|\ell}, \ldots, J^n_{|\ell})$, is the profile of judgments for the first ℓ issues $\{I_1, \ldots, I_\ell\}$.

Let S be a union of issues in A_k; then S defines the *sub-problem* $g_{|S} = (N, S, \neg, \wedge, \mathscr{J} \cap S)$ of the JAP g where $\mathscr{J} \cap S = \{J \cap S | J \in \mathscr{J}\}$.
We proceed now to construct a sequential aggregation function when the issues are ordered.

Definition 3. A tie-breaking rule is a mapping $T : \hat{\mathscr{J}}^N_\ell \to \{p_\ell, \neg p_\ell\}$ for $\ell = 1, \ldots, k$. A tie-breaking rule is *anonymous* if $T(J^{\pi(1)}_\ell, \ldots, J^{\pi(n)}_\ell) = T(J^1_\ell, \ldots, J^n_\ell)$ for any permutation π of $N = \{1, 2, \ldots, n\}$, any $\ell = 1, \ldots, k$ and any profile $J^N_\ell \in \hat{\mathscr{J}}^N_\ell$.

The majority voting with the specific anonymous tie-breaking rule where a tie between p and $\neg p$ is decided in favor of p is called MVAT. It is defined in Section 2 and it is the main ingredient of our aggregation procedure.

Definition 4. Let $g = (N, A_k, \neg, \wedge, \mathscr{J})$ be a JAP with ordered agenda $A_k = (I_1, \ldots, I_k)$. For $\ell = 1, \ldots, k$, let $S_\ell = (I_1 \cup, \ldots, \cup I_\ell)$. A *sequential aggregation function* for g is a sequence of AF's, $(F_1, \ldots, F_k)$, where F_ℓ is an aggregation function of $g_{|S}$ for $\ell = 1, \ldots, k$, such that for every profile $J^N = (J^1, \ldots, J^n)$ and every $\ell = 1, \ldots, k-1$,

$$F_\ell(J^1 \cap S_\ell, \ldots, J^n \cap S_\ell) = F_{\ell+1}(J^1 \cap S_{\ell+1}, \ldots, J^n \cap S_{\ell+1}) \cap S_\ell.$$

Definition 5. Let $g = (N, A_k, \neg, \wedge, \mathscr{J})$ be a JAP with an ordered agenda $A_k = (I_1, \ldots, I_k)$ (that is, $\#A_k = 2k$). The *sequential majority procedure (SMP)* is the sequential aggregation function, F_k, defined inductively on the number of issues, k, as follows.

– For $k = 1$, i.e., $A_1 = \{p_1, \neg p_1\}$, choose between p_1 and $\neg p_1$ by majority with an anonymous tie-breaking rule (MVAT).
– Assume that SMP has been defined for $k \geq 1$ and consider an (ordered) agenda with $k + 1$ (ordered) issues: $A_{k+1} = (\{p_1, \neg p_1\}, \ldots, \{p_k, \neg p_k\}, \{p_{k+1}, \neg p_{k+1}\})$. For a given profile $J^N \in \mathscr{J}^N$, let $F_k(J^N_{|k}) = (q_1, \ldots, q_k)$. Then,

1. If $\{q_1, \ldots, q_k\} \models p_{k+1}$, then SMP chooses p_{k+1} for the $(k+1)$th issue.
2. If $\{q_1, \ldots, q_k\} \models \neg p_{k+1}$, then SMP chooses $\neg p_{k+1}$ for the $(k+1)$th issue.
3. Otherwise, we call $\{p_{k+1}, \neg p_{k+1}\}$ a *free issue*, and SMP chooses from $\{p_{k+1}, \neg p_{k+1}\}$ by MVAT.

Remark 1. Note that the above-defined SMP is indeed a sequential aggregation function according to Definition 4 and that $F_k(J^N)$ is consistent for all $J^N \in \mathscr{J}^N$. This is one of the procedures introduced by Dietrich and List (2007b). We shall provide an axiomatization of SMP (Section 3) and prove that it satisfies an interesting invariance property (III) that is a weak version of proposition-wise independence (PI).

Remark 2. We emphasize that the foregoing SMP depends on the order of introducing the issues of the agenda A_k. Different orderings yield different aggregated judgment, as is the case in the well-known Doctrinal Paradox.

Example 5 (*The Doctrinal Paradox revisited*)**.** The classical example of the Doctrinal Paradox is

	Issues					
	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1

If we apply our SMP with the order of issues $(\{p, \neg p\}, \{q, \neg q\}, \{g, \neg g\})$ we obtain

	Issues					
	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1
SAP(J)	**1**	**0**	**1**	**0**	**1**	**0**

That is, the aggregate judgment is (p, q, g) (in particular, the defendant is liable).

If the order of issues is $(\{p, \neg p\}, \{g, \neg g\}, \{q, \neg q\})$ we obtain

	Issues					
	p	$\neg p$	g	$\neg g$	q	$\neg q$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	0	1	1	0
SAP(J)	**1**	**0**	**0**	**1**	**0**	**1**

That is, the aggregate judgment is $(p, \neg g, \neg q)$ (in particular, the defendant is *not* liable). The same aggregated judgment is obtained for the order $(\{g, \neg g\}, \{p, \neg p\}, \{q, \neg q\})$, while the orders $(\{q, \neg q\}, \{g, \neg g\}, \{p, \neg p\})$ and $(\{g, \neg g\}, \{q, \neg q\}, \{p, \neg p\})$ yield $(q, \neg g, \neg p)$.

We shall argue that in each aggregation problem there is a natural order in which the issues are deliberated. In this example p and then q seem to be the natural temporal order. However, even when the order is given, the aggregation procedure is *vulnerable to manipulation*. For example, in the above-described situation, judge 3 who thinks that the contract is invalid ($\neg p$) and therefore thinks that the defendant is not liable, may dishonestly vote for $\neg q$ in order to reach the verdict "not liable" ($\neg g$).

3. Characterization of SMP

The SMP given in Definition 5 is a sequential aggregation function $(F_1, \ldots, F_k)$ for a JAP $g = (N, A_k, \neg, \wedge, \mathscr{J})$, where $F_\ell : \mathscr{J}^N_{|\ell} \to \mathscr{J}_{|\ell}$ and for $\ell = 1, \ldots, k$, $\mathscr{J}_{|\ell} = \{J \cap (I_1 \cup, \ldots, \cup I_\ell) | J \in \mathscr{J}\}$. (see Definition 4). Thus, *full domain* and *rationality* are guaranteed by definition. Other properties that readily follow from the definition are as follows:

(AN) *Anonymity.*
F_k is anonymous: $F_k(J^{\pi(1)}, \ldots, J^{\pi(n)}) = F_k(J^1, \ldots, J^n)$ for any permutation π of $N = \{1, 2, \ldots, n\}$, and any profile $J^N \in \mathscr{J}^N$ (Recall that MVAT is anonymous).

(RA) *Restricted Agenda*: $F_\ell(J^N_{|\ell}) = F_k(J^N) \cap (I_1 \cup, \ldots, \cup I_\ell)$ for all $J^N \in \mathscr{J}^N$ and all $1 \le \ell \le k$, which follows from the fact that SMP is a sequential aggregation function (Definition 4).

For our characterization of the SMP we introduce the following three properties:

(RM) *Restricted Monotonicity.*
F satisfies restricted monotonicity if for any $i \in N$, $1 \le \ell \le k$, and for any $J^N \in \mathscr{J}^N$ and $\tilde{J}^N \in \mathscr{J}^N$ such that $q^i_\ell = \neg p_\ell$, $\tilde{q}^i_\ell = p_\ell$ and $\tilde{q}^{i'}_{\ell'} = q^{i'}_{\ell'}$ for all $i' \neq i$ or $\ell' \neq \ell$,

$$\text{if } (F(J^N))_\ell = p_\ell \text{ then } (F(\tilde{J}^N))_\ell = p_\ell.$$

That is, if p_ℓ is chosen by F and then only one voter switches from $\neg p_\ell$ to p_ℓ (while keeping a consistent judgment), then p_ℓ will be chosen by F also in the modified profile of judgments.

(IPD) *Independence of Past Deliberations.*
F satisfies independence of past deliberations if for all $1 \le \ell < k$ and for any profiles J^N and $\tilde{J}^N$,

$$\text{if } F_\ell(J^N_{|\ell}) = F_\ell(\tilde{J}^N_{|\ell}) \text{ and } J^N_{\ell+1} = \tilde{J}^N_{\ell+1} \text{ then}$$
$$(F_k(J^N))_{\ell+1} = (F_k(\tilde{J}^N))_{\ell+1}.$$

That is, the collective judgment on issue $I_{\ell+1}$ depends only on the individual judgments on this issue and the collective judgments on the preceding issues $I_1, \ldots, I_\ell$ (and not on the profile of judgments on these issues).

To define the last axiom we introduce some notation: For an ℓ-judgment $J_{|\ell}$, denote by $(J_{|\ell})^N = (J_{|\ell}, \ldots, J_{|\ell})$ the ℓ-profile in which all judges have the same ℓ-judgment $J_{|\ell}$. Given $J^N_{\ell+1} = (q^1_{\ell+1}, \ldots, q^n_{\ell+1})$, a profile of judgments on issue $I_{\ell+1}$, denote by $\neg J^N_{\ell+1} = (\neg q^1_{\ell+1}, \ldots, \neg q^n_{\ell+1})$ its (componentwise) negation and by $\overrightarrow{J^N_{\ell+1}}$ the profile $J^N_{\ell+1}$ ordered with all $p_{\ell+1}$ first and then $\neg p_{\ell+1}$, that is, $\overrightarrow{J^N_{\ell+1}} = (p_{\ell+1}, \ldots, p_{\ell+1}, \neg p_{\ell+1}, \ldots, \neg p_{\ell+1})$. Our last axiom is now stated as follows:

(LN) *Limited Neutrality.* The aggregation function F satisfies limited neutrality if for all $1 \le \ell < k$ and all $J^N \in \mathscr{J}^N$, if both $(F_\ell(J^N_{|\ell}), p_{\ell+1})$ and $(F_\ell(J^N_{|\ell}), \neg p_{\ell+1})$ are consistent, then

$$F_{\ell+1}((F_\ell(J^N_{|\ell})^N, \neg J^N_{\ell+1}))_{\ell+1} = \begin{cases} p_{\ell+1} & \text{if } \overrightarrow{\neg J^N_{\ell+1}} = \overrightarrow{J^N_{\ell+1}} \\ \neg((F_\ell(J^N_{|\ell})^N, J^N_{\ell+1}))_{\ell+1} & \text{otherwise.} \end{cases}$$

In words, limited neutrality requires neutrality between $p_{\ell+1}$ and $\neg p_{\ell+1}$ (with tie-breaking rule in favor of $p_{\ell+1}$) only when there is unanimity of judgments on previous issues and when both $p_{\ell+1}$ and $\neg p_{\ell+1}$ are consistent with these unanimous judgments.

Remark 3. It follows from the definition that SMP also satisfies the following properties which are not used in our axiomatization:

(U) *Unanimity.*
F_k is unanimous: $F_k(J, \ldots, J) = J$ for all $J \in \mathscr{J}$.

(REIN) *Reinforcement* (see Section 5).

Remark 4. In Section 4 we prove (Proposition 3) that the SMP satisfies *Independence of irrelevant issues* (III) with respect to the relevance relation R^* given in Definition 10. This is a weak version of the *Independence of irrelevant propositions* (IIP) which plays a central role in deriving impossibility results.

In preparation for our main characterization theorem we first characterize the aggregation procedure for the case of a single issue ($k = 1$) by modifying May's (1952) axiomatization of the majority rule. While May's model allows for the neutrality between two alternatives, in our model, the choice is between a proposition and its negation that must be single-valued, and no neutrality is possible (in May's notation the values of the decision function are in $\{-1, 1\}$ rather than $\{-1, 0, 1\}$.)

We consider the case of $N = \{1, \ldots, n\}$ voters and two alternatives, p and $\neg p$. Each voter chooses one alternative. *Majority voting with anonymous tie-breaking* (MVAT) is defined as follows:

- If n is odd then the majority alternative is selected by the group.
- If $n = 2k$ and exactly k members choose p, then p is chosen; otherwise, the majority alternative is chosen.

That is, MVAT for the issue $(p, \neg p)$ is a majority voting where tie is decided in favor of p.

Denote $d(i) = 1$ if voter i chooses p, and $d(i) = -1$ if voter i chooses $\neg p$. Let $d = (d(1), \ldots, d(n))$. A *voting rule* (VR) is a function $f : \{1, -1\}^N \to \{1, -1\}$. Obviously, MVAT can be written as a voting rule. It satisfies the following axioms.

(AN*) *Anonymity*. $f(d(1), \ldots, d(n)) = f(d(\pi(1)), \ldots, d(\pi(n)))$ for all permutations π of N.

(M*) *Monotonicity*. $[d(i) = d^*(i)\ \forall i \neq j$, and $d(j) > d^*(j)] \models f(d) \ge f(d^*)$.

(LN*) *Limited neutrality*. For all n and all d, if $|\{i : d(i) = 1\}| = |\{i : d(i) = -1\}|$, then $f(d) = f(-d) = 1$; otherwise, $f(-d) = -f(d)$.

Although these properties are closely related to the above defined properties for aggregation functions, we added the asterisk (*) to indicate that they relate to a different model.

Theorem 1. *There is a unique VR f that satisfies (AN*), (M*), and (LN*) and it is MVAT.*

Proof. This is actually a slight modification of May's characterization but it can be directly proved as follows. Call a coalition of voters "winning" if when all its members vote 1 then society's vote is also 1. This defines a *simple game* which, by anonymity and monotonicity, is the simple game (n, k) where k is in $\{0, \dots, n\}$, that is, a game in which a coalition is winning if and only if it has at least k members. Limited neutrality (LN*) now yields that $k = (n+1)/2$ if n is odd and $k = n/2$ with tie breaking rule in favor of 1, which is MVAT. ■

We are now ready to state our characterization theorem for SMP.

Theorem 2. *There is one and only one aggregation function F satisfying the axioms (AN), (RA), (RM), (IPD), and (LN). It is the sequential majority procedure (SMP).*

Proof. It follows from Definition 5 that SMP satisfies all five axioms. Let F be a judgment aggregation function satisfying the axioms. We shall prove that it is SMP. The idea is that at any issue I_ℓ; $\ell = 1, \dots, k$, if the choice is implied by consistency with previous choices, it must be the same choice for F and for SMP since both are consistent procedures. Otherwise issue I_ℓ is a 'free issue'. We shall apply Theorem 1 to prove that F has to choose between p_ℓ and $\neg p_\ell$ by MVAT, just like SMP. For that we have to show that the voting rule VR used by F in a free issue satisfies (AN*), (M*), and (LN*). Indeed, these follow from (AN), (RM), (IPD), and (LN):

- (AN) $\Rightarrow$ (AN*) on free issues.
- (RM) and (IPD) $\Rightarrow$ (M*) on free issues.
  Note that the (IPD) is needed for this implication since (RM)$\not\Rightarrow$ (M*) on free issues. The reason is (RM) considers switches from $\neg p_{\ell+1}$ to $p_{\ell+1}$ only for players whose judgment remains consistent after this switch while in (M*) this switch is allowed for all players. Using (IPD) every player's judgment $J^i_{|\ell}$, can be replaced by $F_\ell(J^N_{|\ell})$ which, when $I_{\ell+1}$ is a free issue, is consistent with both $\neg p_{\ell+1}$ and $p_{\ell+1}$.
- (LN) and (IPD) $\Rightarrow$ (LN*) on free issues.
  Here again the (IPD) is needed since it is implicitly used in the definition of (LN).

We conclude that since F satisfies (AN), (RM), (LN) and (IPD), the voting rule in any free issue satisfies (AN*), (M*) and (LN*) and hence, by Theorem 1, in every free issue, F chooses a proposition by MVAT. We proceed now to prove that F must coincide with SMP:

- Since F satisfies the *restricted agenda* property (RA), F is sequential and we have to show that for each issue I_ℓ (formally by induction of ℓ) F coincides with SMP.
- For $k = 1$, axioms (AN), (RM), and (LN) lead, by Theorem 1, to majority voting with an anonymous tie-breaking rule (MVAT) in favor of p_1.
- Assume that F coincides with SMP for an agenda of up to k issues and let us prove it for the $(k+1)$th issue. Given a profile J^N with $k+1$ issues:
  - If $F_{|k}(J^N) \models p_{k+1}$ or $F_{|k}(J^N) \models \neg p_{k+1}$, then by consistency $F_{k+1}(J^N) = p_{k+1}$ or $F_{k+1}(J^N) = \neg p_{k+1}$, respectively, and hence F coincides with SMP on the $(k+1)$th issue.
  - Otherwise both $(F_{|k}(J^N), p_{k+1})$ and $(F_{|k}(J^N), \neg p_{k+1})$ are consistent.

  By the (IPD) axiom, $F(J^N) = F_{k+1}((F_{|k}(J^N))^N, J^N_{k+1})$ and again (as for $k = 1$), by (AN), (RM), and (LN), this implies that the $(k+1)$th issue is decided by MVAT, as in SMP, completing the proof. ■

Remark 5. Note that when we applied (in Theorem 2) the MVAT à la May, we had full domain, both of J^N_1 for the first step $k = 1$ and of J^N_{k+1} in the induction step.

Independence of the axioms

For each of the five axioms we show an aggregation function not satisfying that axiom but satisfying all four other axioms.

(AN) Dictatorship when n is odd and $n \geq 3$, satisfies all axioms except (AN).
Actually, when n is even the axiom (AN) is redundant since it is implied by the other four axioms. Indeed, consider a free issue I_h. If $|\{i : J^i_h = p_h\}| = n/2$, then p_h is chosen. By (IPD), (RM) and (LN), any coalition S of voters of cardinality $|S| \geq n/2 + 1$ is effective for (i.e. can impose) both p_h and $\neg p_h$. Thus in any free issue I_h, the choice between p_h and $\neg p_h$ depends only on the number of votes for each proposition, hence the anonymity.

(RA) Let σ^* be the permutation of the issues $\{I_1, \dots, I_K\}$ given by $\sigma^*(I_k) = I_{K-k+1}$, for $k = 1, \dots, K$. Let F be SMP and consider the following aggregation function F^* defined by

$$F^*(J^N) = F(\sigma^*(J^N))$$

where $\sigma^*(J^N)$ is obtained from the profile J^N by reordering the issues according to the permutation σ^*. The function F^* satisfies (AN), (LN), (RM), and (IPD) since SMP, F, satisfy these axioms. However, F^* does not satisfy (RA) as can be seen in the following Doctrinal Paradox:
Considering the three issues $\{(p, \neg p), (q, \neg q), (g, \neg g)\}$ with $g \Leftrightarrow p \wedge q$ and the judgment profile J^N of three judges given by

	Issues					
	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1

Then, $F^*(J^N) = (\neg p, q, \neg g)$ but $F^*(J^N$ restricted to $\{(p, \neg p), (q, \neg q)\}) = (p, q)$.

(LN) Let F be our SMP and let $\tilde{F}$ be the same procedure except that for a free issue $(p_k, \neg p_k)$,

$$\tilde{F}_k(J^N) = \begin{cases} \neg p_k & \text{if } |\{i|J^i_k = \neg p_k\}| > \dfrac{2}{3}n \\ p_k & \text{otherwise.} \end{cases}$$

This $\tilde{F}$ satisfies all axioms except (LN).

(RM) Let F be our SMP and let $\hat{F}$ be the same procedure except that a free issue, $(p_k, \neg p_k)$, is decided by *"minority principle"* that is, the chosen proposition is the one supported by a minority (with anonymous tie-breaking rule). This $\hat{F}$ satisfies all axioms except (RM) ($\hat{F}$ satisfies *anti-monotonicity*).

(IPD) Consider F^*, which is the same as SMP except that for a free issue $(p_{k+1}, \neg p_{k+1})$ for $k \geq 1$, the decision is made as follows:

- $F^*_{k+1}(J^N) = p_{k+1}$ If $|\{i|J^i_{|k} \models p_{k+1}\}| > |\{i|J^i_{|k} \models \neg p_{k+1}\}|$.
- $F^*_{k+1}(J^N) = \neg p_{k+1}$ if $|\{i|J^i_{|k} \models \neg p_{k+1}\}| > |\{i|J^i_{|k} \models p_{k+1}\}|$.
- Otherwise, $F^*_{k+1}(J^N) \in \{p_{k+1}, \neg p_{k+1}\}$ is chosen by simple majority (with anonymous tie-breaking rule).

This F^* satisfies all axioms except (IPD). (To see that F^* satisfies (RM) note that when a judge changes his judgment on issue $(p_{k+1}, \neg p_{k+1})$ from $\neg p_{k+1}$ to p_{k+1}, it affects only the third possibility above, where monotonicity is clear.)

4. Relevance relations: From IIA to III

The most crucial axiom in Arrow's impossibility theorem is IIA — *independence of irrelevant alternatives*. The analogue axiom for judgment aggregation would be PI — *proposition-wise independence*. It turns out that this axiom is too strong and, together with a few mild assumptions, it readily yields impossibility results (see, e.g., List, 2012). Any attempt to obtain positive results must go through weakening this axiom. Such a weakening was suggested by Dietrich (2015) who replaced PI by IIP — *independence of irrelevant propositions*, with respect to an abstract given relevance relation. We adopt this idea but attempt to derive the *relevance relation* from the agenda: we will derive a "natural" *relevance relation* between propositions in the agenda and show that our proposed aggregation function satisfies IIP. We first recall that Dietrich assumed that the (abstract) relevance relation R between propositions satisfies two conditions (we adopt Dietrich's notation and write $\{\pm p\}$ for $\{p, \neg p\}$):

- *Negation-invariance* (Dietrich, 2015 Equation (1), p. 470):

$$qRp \Leftrightarrow q'Rp' \text{ if } q' \in \{\pm q\} \text{ and } p' \in \{\pm p\}.$$

We notice that a relation R satisfying negation invariance is actually a relation between *issues*; therefore, we will adopt this terminology and define a relevance relation between the issues of the agenda $A = \{I_1, \ldots, I_k\}$.

Definition 6. A *relevance relation* R is a reflexive and acyclic binary relation between the issues of the agenda A. "I_j is relevant to I_h" is denoted by I_jRI_h and for $I_h \in A$, the set $R(I_h) = \{I_j | I_jRI_h\}$ is the set of issues relevant to issue I_h. For convenience, when no confusion may arise, we use the same notation for the set of propositions in these issues, i.e.,

$$R(I_h) = \cup\{p_j, \neg p_j\} : I_j = \{p_j, \neg p_j\}RI_h.$$

The analogue of the IIA axiom is the III axiom (*independence of irrelevant issues*) defined as follows.

Definition 7 (*Independence of Irrelevant Issues (III)*)**.** Given a JAP, $g = (N, A_k, \neg, \wedge, \mathscr{J})$, a judgment aggregation function $F : \mathscr{J}^N \to \mathscr{J}$ satisfies *independence of irrelevant issues* (III) w.r.t. the relevance relation R, if for all $J_1^N, J_2^N \in \mathscr{J}^N$, and for all $I_h \in A$,

$$\left[J_1^i \cap R(I_h) = J_2^i \cap R(I_h),\ \forall i \in N,\ \text{and } p^* \in I_h\right]$$
$$\Longrightarrow \left[p^* \in F(J_1^N) \Leftrightarrow p^* \in F(J_2^N)\right].$$

Example. If $R(I_h) = \{I_h\}$ for all $I_h \in A$, then for $p^* \in I_h$,

$$\left[J_1^i \cap R(I_h) = J_2^i \cap R(I_h),\ \forall i \in N\right]$$
$$\Leftrightarrow \left[p^* \in J_1^i \Leftrightarrow p^* \in J_2^i;\ \forall i \in N; \forall p^* \in I_h\right],$$

and III is equivalent in this case to proposition-wise independence (PI).

The first natural attempt to derive a relevance relation from the agenda is

Definition 8 (*Relevance by Direct Entailment*)**.** Given a JAP, $g = (N, A_k, \neg, \wedge, \mathscr{J})$, and a fixed order of the agenda: $A_k = \{I_1, \ldots, I_k\}$, the relevance relation EM (*entailment*) is a correspondence $EM : A_k \to 2^{A_k}$ defined by,

$$I_j \in EM(I_h) \text{ if } j \le h \text{ and } [\exists\, q^* \in I_j \text{ and } \exists p^* \in I_h \text{ such that } q^* \models p^*].$$

When $p \in I_h$ we also write $EM(p)$ for $EM(I_h)$.

Remark 6. We note that

1. This relevance relation is *reflexive* $(I_h \in EM(I_h))$; $\forall I_h \in A$, but it is *not transitive*.
2. This relevance relation is *not symmetric*; that is, I_jRI_h does not imply I_hRI_j. Furthermore, for $j \neq h$, if I_jRI_h then I_hRI_j cannot hold even if $p^* \models q^*$ for some $q^* \in I_j$ and $p^* \in I_h$ since $j \le h$ excludes $h \le j$ for $j \neq h$. In other words, the issue I_h is irrelevant to the issue I_j even if there is a logical implication since it is decided *after* I_j.

Nevertheless, for the case of two issues we have the following:

Proposition 1. *For $k = 1,2$, the aggregation function F, given in Definition 2, satisfies independence of irrelevant issues (III) w.r.t. the relevance relation EM defined by Definition 8.*

Proof. We have to prove that for each $j \le k$, $p \in \{p_j, \neg p_j\}$, and all $J_1^N, J_2^N \in \mathscr{J}^N$,

$$J_1^i \cap EM(p) = J_2^i \cap EM(p),\ \forall i \in N \Longrightarrow \left[p \in F(J_1^N) \Leftrightarrow p \in F(J_2^N)\right].$$

1. For $k = 1$, $A_1 = (I_1) = \{p, \neg p\}$ and $EM(p) = \{I_1\} = \{p, \neg p\}$. By our assumption $p \in J_1^i$ if and only if $p \in J_2^i$ for all $i \in N$. As p is admitted to the collective choice set by majority rule, $p \in F(J_1^N)$ if and only if $p \in F(J_2^N)$.
2. For $k = 2$, $A_2 = (I_1, I_2) = (\{p_1, \neg p_1\}, \{p_2, \neg p_2\})$. By part 1., we have only to consider the second issue.
   Let $p \in \{p_2, \neg p_2\}$. We distinguish the following cases:
   - 2.1 $EM(p) = \{I_2\}$ (and thus $EM(\neg p) = \{I_2\}$). Then $F(J_1^N)$ and $F(J_2^N)$ are determined by majority rule. As $EM(p) = \{I_2\} = \{p, \neg p\}$ and $p \in J_1^i$ if and only if $p \in J_2^i$ for all $i \in N$, it follows that $p \in F_2(J_1^N)$ if and only if $p \in F_2(J_2^N)$.
   - 2.2 There is $q \in \{p_1, \neg p_1\}$ such that $q \models p$. By our assumptions $q \in J_1^i$ if and only if $q \in J_2^i$, $F_1(J_1^N) = F_1(J_2^N)$ (as in part 1.) Then there are two possibilities:
     - If $F_1(J_1^N) = F_1(J_2^N) = q$ (i.e., q has majority), then by entailment, $F_2(J_1^N) = F_2(J_2^N) = p$.
     - If $F_1(J_1^N) = F_1(J_2^N) = \neg q$ (i.e., $\neg q$ has majority) then,
       - $\neg q \not\models p$, since $\neg q \models p$ (and $q \models p$) implies that p is a tautology.
       - $\neg q \not\models \neg p$ since $\neg q \models \neg p$ (and $q \models p$) implies that q and p are equivalent.
       - Therefore, $F_2(J^N)$ is determined by majority voting and we obtain again $F_2(J_1^N) = F_2(J_2^N)$ (since for all $i \in N$; $p \in J_1^i \Leftrightarrow p \in J_2^i$).
   - 2.3 There is $q \in \{p_1, \neg p_1\}$ such that $q \models \neg p$. This is treated similarly to the previous case 2.2.
   This completes the proof. ■

Unfortunately, Proposition 1 cannot be extended to $k > 2$. Furthermore, the following example shows that for $k > 2$, our aggregation function SMP cannot satisfy III w.r.t. any relevance relation between two propositions based only on binary implications between the propositions or their negations.

Example 6. Consider the following the agenda with three issues $A_3 = \{I_1, I_2, I_3\}$ corresponding to the following three propositions and their negations (put in the semantic setting[1]):

$\Omega = \{a_1, a_2, \dots, a_8\}$

$p_1 = \{a_1, a_2, a_5, a_6\}$ $\neg p_1 = \{a_3, a_4, a_7, a_8\}$

$p_2 = \{a_1, a_3, a_7, a_8\}$ $\neg p_2 = \{a_2, a_4, a_5, a_6\}$

$p_3 = \{a_1, a_4, a_7, a_8\}$ $\neg p_3 = \{a_2, a_3, a_5, a_6\}$

First, observe that there is no entailment relation between any two of the propositions and their negations; that is, $EM(I_j) = \{I_j\}$ for $j = 1, 2, 3$. Next we see that $p_1 \wedge p_2 \models p_3$, $\neg p_1 \wedge \neg p_2 \models p_3$, and $p_1 \wedge \neg p_2 \models \neg p_3$.

For the order of issues (I_1, I_2, I_3) our aggregation function yields

$$F\left((p_1, p_2, p_3), (p_1, \neg p_2, \neg p_3), (\neg p_1, p_2, p_3)\right) = (p_1, p_2, p_3),$$

as p_1 and p_2 are decided by majority rule and I_3 is determined by $p_1 \wedge p_2 \models p_3$.

Changing p_2 in the judgment of the third voter to $\neg p_2$ yields

$$F\left((p_1, p_2, p_3), (p_1, \neg p_2, \neg p_3), (\neg p_1, \neg p_2, p_3)\right) = (p_1, \neg p_2, \neg p_3),$$

since p_1 and $\neg p_2$ are decided by majority rule and then I_3 is determined since $p_1 \wedge \neg p_2 \models \neg p_3$. This contradicts III since I_2 is irrelevant to I_3.

In view of our last example, if our objective is to have our aggregation function F satisfy III, we must introduce a relevance relation of a wider range than that of simple implication.

Definition 9. Let $j \le h$, $h > 1$. The issue I_j is relevant to the issue I_h (notation $I_j R I_h$) if there exist $p \in I_h$, $q \in I_j$, and a set of issues $(I_\ell)_{\ell \in L}$, where $L \subset \{1, \dots, h-1\}$ (which may be empty), and $q_\ell \in I_\ell$, $\ell \in L$ such that the set $S = \{q_\ell | \ell \in L\}$ satisfies the following requirements:

$$S \cup q \quad \text{is consistent} \tag{1}$$

$$S \cup q \models p \tag{2}$$

$$S \not\models p \tag{3}$$

Interpretation Denoting by $\mathscr{J}_h$ the set of all rational judgments of the issues $(I_1, \dots, I_h)$, for distinct issues $(j < h)$, the intuition formalized in this definition is that the issue $\{\pm q\}$ is relevant to proposition $\{\pm p\}$ if the following conditions hold:

1. The issue $\{\pm q\}$ is decided (appears in our given order) before the issue $\{\pm p\}$.
2. All $J \in \mathscr{J}_h$ satisfy $S \cup q \subset J \Rightarrow p \in J$. ($S \cup q \models p$.)
3. $\exists J^* \in \mathscr{J}_h$ such that $S \cup \neg p \subset J^*$. ($S \not\models p$.)

Remark 7. Note that R is reflexive: $p \in R(p)$ (by $p \models p$). Also, for $L = \emptyset$ (hence $S = \emptyset$), the conditions (1), (2), (3) reduce to straight entailment $q \models p$, and hence the relevance relation R is an extension of the implication relation; that is, $EM(p) \subset R(p)$ for all propositions $p \in A$.

Remark 8. This relevance relation is very closely related to the notion of *conditional entailment* introduced first by Nehring and Puppe and then defined again by Dietrich and List: "q conditionally entails p (denoted by $q \models^* p$) if there is $S \subseteq A$ that is consistent both with q and with $\neg p$ such that $S \cup \{q\} \models p$" (see Dietrich and List, 2008, p. 21.) The relation to the relevance relation R in Definition 9 is as follows: the issue I_j is relevant to the issue I_h if $j < h$ and there exist $p \in I_h$, and $q \in I_j$ such that q conditionally entails p (i.e., $q \models^* p$) with S being a subset of propositions preceding p (that is $S = \{q_\ell | \ell \in L\}$ where $L \subset \{1, \dots, h-1\}$).

[1] In all our examples using a finite semantic logic, we take $\mathscr{J}$ to be the set of all complete and consistent (i.e., with nonempty intersection) judgments.

The relevance relation in Definition 9 is *not transitive* as is demonstrated by the following example presented in the semantic setting.

Example 7. Let $W = \{a, b, c, d, e, f, g, h, m\}$ and consider the following issues (I_1, I_2, I_3, I_4), where $I_j = \{q_j, \neg q_j\}, j = 1, 2, 3, 4$, with the propositions:

$q_1 = \{a, b\}$ $\neg q_1 = \{c, d, e, f, g, h, m\}$
$q_2 = \{c, d, e\}$ $\neg q_2 = \{a, b, f, g, h, m\}$
$q_3 = \{a, b, c, f, g\}$ $\neg q_3 = \{d, e, h, m\}$
$q_4 = \{a, c, g, h\}$ $\neg q_4 = \{b, d, e, f, m\}$

With respect to our relevance relation (Definition 9), we have

- $q_1 \models q_3$ (and $q_1 \models \neg q_2$), and hence $q_1 \in R(q_3)$ (and $q_1 \in R(q_2)$).
- $q_2 \wedge q_3 \models q_4$, $q_2 \not\models q_4$ and $q_3 \not\models q_4$, and hence $q_3 \in R(q_4)$ (and $q_2 \in R(q_4)$).

We claim that q_1 is not relevant to q_4. Indeed:

- $q_1 \wedge \neg q_2 = q_1 \not\models q_4$ (or$\neg q_4$) ($q_1 \not\models I_4$ for short).
- $q_1 \wedge q_3 = q_1 \not\models I_4$ and $\neg q_1 \wedge q_2 = q_2 \not\models I_4$.
- $\neg q_1 \wedge \neg q_2 = \{f, g, h, m\} \not\models I_4$, and $\neg q_1 \wedge q_3 = \{c, f, g\} \not\models I_4$.
- Finally, $\neg q_1 \wedge \neg q_3 = \{d, e, h, m\} \not\models I_4$, completing the check of all pairs of propositions including q_1.

We proceed checking all triples of propositions including q_1:

- $q_1 \wedge q_2 = \emptyset$, eliminating the two triples $q_1 \wedge q_2 \wedge q_3$ and $q_1 \wedge q_2 \wedge \neg q_3$.
- $q_1 \wedge \neg q_2 = q_1$, eliminating the two triples $q_1 \wedge \neg q_2 \wedge q_3$ and $q_1 \wedge \neg q_2 \wedge \neg q_3$, by our results for pairs.
- $\neg q_1 \wedge q_2 \wedge q_3 \models q_4$ and $\neg q_1 \wedge q_2 \wedge \neg q_3 \models \neg q_4$; however, in both cases $\neg q_1$ is redundant for the entailment and therefore it does not satisfy the conditions for relevance to q_4 or $\neg q_4$.

The remaining two triples to check are as follows:

- $\neg q_1 \wedge \neg q_2 \wedge q_3 = \{f, g\} \not\models I_4$.
- Finally, $\neg q_1 \wedge \neg q_2 \wedge \neg q_3 = \{h, m\} \not\models I_4$.

This completes the proof that $I_1 \notin R(I_4)$, and hence this relevance relation is *not transitive*.

The following proposition will be used in our proofs in the sequel.

Proposition 2. *For any $p \in I_h$ and any restricted consistent judgment $J_{|h-1}$, the following holds:*

$$J_{|h-1} \models p (\text{or } \neg p) \text{ if and only if } J_{|h-1} \cap R(p) \models p (\text{or } \neg p).$$

Proof. The "if" part follows since $J_{|h-1} \cap R(p) \subset J_{|h-1}$ (by the monotonicity of the entailment).

For the "only if" part assume that $J_{|h-1} \models p(\text{or } \neg p)$ and $J_{|h-1} \cap R(p) \not\models p(\text{or } \neg p)$. If the propositions in $J_{|h-1} \setminus R(p)$ are removed one by one from $J_{|h-1}$, there must be a first case in which, when $\tilde{q} \notin R(p)$ is removed, the entailment $\models p$ (or $\models \neg p$) no longer holds. Taking in Definition 9 the set $S \subseteq J_{|h-1}$ to be the set of propositions not removed up to that stage (after removing $\tilde{q}$), we have that $\tilde{q} \in R(p)$ in contradiction to $\tilde{q} \in J_{|h-1} \setminus R(p)$. ■

Although the transitivity of our relevance relation is not required for the previous proposition, it seems to be necessary for the III property of SMP as is demonstrated by the following example (built on Example 7) in which III is violated.

Example 8 (*Violation of III*). Let $W = \{a, b, c, d, e, f, g, h, m\}$, $W' = \{a', b', c', d', e', f', g', h', m'\}$, and $\Omega = W \cup W'$. Let q_1, q_2, q_3, q_4

be the following subsets of W (and their complements), defined in Example 7:

$q_1 = \{a, b\}$ $\quad q_1^c = \{c, d, e, f, g, h, m\}$
$q_2 = \{c, d, e\}$ $\quad q_2^c = \{a, b, f, g, h, m\}$
$q_3 = \{a, b, c, f, g\}$ $\quad q_3^c = \{d, e, h, m\}$
$q_4 = \{a, c, g, h\}$ $\quad q_4^c = \{b, d, e, f, m\}$

For $k = 1, \ldots, 4$, let q'_k be the subset of W' defined by $q'_k = \{w' \in W' | w \in q_k\}$ and consider the following five propositions (subsets) in Ω:

$$q_{10} = q_1 \cup W',\ q_{01} = W \cup q'_1,\ q_{kk} = q_k \cup q'_k,\ \ k = 2, 3, 4,$$

and the corresponding five issues:

$$I_{10} = \{q_{10}, \neg q_{10}\},\ I_{01} = \{q_{01}, \neg q_{01}\},\ \ I_{kk} = \{q_{kk}, \neg q_{kk}\},\ k = 2, 3, 4.$$

Considering the agenda of five (ordered) issues, $A = (I_{10}, I_{01}, I_{22}, I_{33}, I_{44})$, we have the following:

- $I_{10} \wedge I_{01} \models q_{33}$, $I_{01} \not\models q_{33}$ (and $I_{10} \not\models q_{33}$), hence $q_{10} \in R(I_{33})$ (and $q_{01} \in R(I_{33})$).
- $I_{22} \wedge I_{33} \models q_{44}$, $I_{22} \not\models q_{44}$ (and $I_{33} \not\models q_{44}$), hence $q_{33} \in R(I_{44})$ (and $q_{22} \in R(I_{44})$).

Claim 1. *The issue I_{10} is not relevant to the issue I_{44}, that is, $I_{10} \notin R(I_{44})$ (non-transitivity).*

Proof. See Appendix.

Assume that three judges debating the five issues presented above, have the following profile of judgments J_1^N:

	Issues	I_{10}	I_{01}	I_{22}	I_{33}	I_{44}
J_1^N	Judge 1	q_{10}	q_{01}	$\neg q_{22}$	q_{33}	$\neg q_{44}$
	Judge 2	$\neg q_{10}$	q_{01}	q_{22}	$\neg q_{33}$	$\neg q_{44}$
	Judge 3	q_{10}	$\neg q_{01}$	q_{22}	q_{33}	q_{44}

Aggregation of these judgments according to SMP yields the following:

	Issues	I_{10}	I_{01}	I_{22}	I_{33}	I_{44}
J_1^N	Judge 1	q_{10}	q_{01}	$\neg q_{22}$	q_{33}	$\neg q_{44}$
	Judge 2	$\neg q_{10}$	q_{01}	q_{22}	$\neg q_{33}$	$\neg q_{44}$
	Judge 3	q_{10}	$\neg q_{01}$	q_{22}	q_{33}	q_{44}
	SMP	$\mathbf{q_{10}}$	$\mathbf{q_{01}}$	$\mathbf{\neg q_{22}}$	$\mathbf{q_{33}}$	$\mathbf{\neg q_{44}}$

(q_{10}, q_{01} are obtained by majority voting, $q_{10} \wedge q_{01} \models \neg q_{22}$, $q_{10} \wedge q_{01} \models q_{33}$, and $\neg q_{44}$ is obtained by majority voting).

Consider now the following profile of judgments J_2^N, which differs from the profile J_1^N only by Judge 1 switching opinion on issue I_{10} (which is irrelevant to I_{44}), from q_{10} to $\neg q_{10}$:

	Issues	I_{10}	I_{01}	I_{22}	I_{33}	I_{44}
J_2^N	Judge 1	$\neg q_{10}$	q_{01}	$\neg q_{22}$	q_{33}	$\neg q_{44}$
	Judge 2	$\neg q_{10}$	q_{01}	q_{22}	$\neg q_{33}$	$\neg q_{44}$
	Judge 3	q_{10}	$\neg q_{01}$	q_{22}	q_{33}	q_{44}
	SMP	$\mathbf{\neg q_{10}}$	$\mathbf{q_{01}}$	$\mathbf{q_{22}}$	$\mathbf{q_{33}}$	$\mathbf{q_{44}}$

($\neg q_{10}, q_{01}, q_{22}, q_{33}$ are obtained by majority voting and then $\neg q_{10} \wedge q_{01} \wedge q_{22} \wedge q_{33} \models q_{44}$).

As I_{10} is irrelevant to I_{44}, this is in contradiction to the III property.

In view of Example 8 we take the transitive closure of our relevance relation.

Definition 10. The relevance relation R^* is the *transitive closure* of the relevance relation R given in Definition 9.

Since $R^*(p) \supseteq R(p)$ for all propositions p, Proposition 2 clearly holds also for the relevance relation R^* and we have the following corollary:

Corollary 1. *For any $p \in I_h$ and any restricted consistent judgement $J_{|h-1}$ the following holds:*

$$J_{|h-1} \models p(\text{or } \neg p) \ \text{ if and only if } \ J_{|h-1} \cap R^*(p) \models p(\text{or } \neg p)$$

Proposition 3. *Our aggregation function F (SMP), given in Definition 2, satisfies III w.r.t. the relevance relation R^* given in Definition 10.*

Proof. Let $J_1^N, J_2^N \in \mathscr{J}^N$, and let $p \in I_h$. We have to prove that if $J_1^i \cap R^*(p) = J_2^i \cap R^*(p)$ for all $i \in N$; then $p \in F(J_1^N)$ if and only if $p \in F(J_2^N)$. Actually we will prove a stronger result. Namely, under the same conditions $F(J_1^N) \cap R^*(p) = F(J_2^N) \cap R^*(p)$; that is, not only does $p \in F(J_1^N)$ if and only if $p \in F(J_2^N)$ but also $q \in F(J_1^N)$ if and only if $q \in F(J_2^N)$ for all $q \in R^*(p)$. In other words, if $J_1^i \cap R^*(p) = J_2^i \cap R^*(p)$ for all $i \in N$, then not only the appearance of p is the same in both $F(J_1^N)$ and $F(J_2^N)$ but this is true for all propositions relevant to p.

The proof is by induction on h. The case $h = 1$ follows from our assumptions, the reflexivity of $R^*(\cdot)$, and the definition of F. Let $h > 1$ and assume by induction that the claim is true for $j = 1, \ldots, h-1$.

Note first that from the transitivity of R^* we have $q \in R^*(p) \Rightarrow R^*(q) \subset R^*(p)$ and therefore from

$$J_1^i \cap R^*(p) = J_2^i \cap R^*(p),\ \forall i \in N$$

we also have (by intersecting both sides with $R^*(q)$),

$$J_1^i \cap R^*(q) = J_2^i \cap R^*(q),\ \forall i \in N,\ \forall q \in R^*(p)$$

and therefore by the induction hypothesis,

$$F(J_1^N) \cap R^*(q) = F(J_2^N) \cap R^*(q),\ \ \forall q \in I_j,\ j < h,\ q \in R^*(p),$$

and hence

$$(F(J_1^N))_{|h-1} \cap R^*(p) = (F(J_2^N))_{|h-1} \cap R^*(p). \tag{4}$$

We distinguish two cases.

1. If $(F(J_1^N))_{|h-1} \models p$. In this case, it must also be that $(F(J_2^N))_{|h-1} \models p$.
   Indeed, by Corollary 1 we have $(F(J_1^N))_{|h-1} \cap R^*(p) \models p$ and, by Eq. (4), $(F(J_2^N))_{|h-1} \cap R^*(p) \models p$. Applying Corollary 1 again we have $(F(J_2^N))_{|h-1} \models p$.
   Similarly, if $(F(J_1^N))_{|h-1} \models \neg p$ then also $(F(J_2^N))_{|h-1} \models \neg p$.
   It follows that in this case SMP chooses p (or $\neg p$) in both J_1^N and J_2^N. Combining this with Eq. (4), we get $F(J_1^N) \cap R^*(p) = F(J_2^N) \cap R^*(p)$.
2. If $(F(J_1^N))_{|h-1} \not\models p$ and $(F(J_1^N))_{|h-1} \not\models \neg p$, then again by Corollary 1 and Eq. (4) (by the same argument as in part 1.) we also have $(F(J_2^N))_{|h-1} \not\models p$ and $(F(J_2^N))_{|h-1} \not\models \neg p$. Hence the issue $\{p, \neg p\}$ is decided by simple majority voting in both profiles. Since for all $i \in N$, $p \in J_1^i$ if and only if $p \in J_2^i$, we get $p \in F(J_1^N)$ if and only if $p \in F(J_2^N)$. Combining this with Eq. (4) we get $F(J_1^N) \cap R^*(p) = F(J_2^N) \cap R^*(p)$, completing the proof. ■

5. Choice by plurality voting (CPV)

Definition 11. Let $g = (N, A_k, \neg, \wedge, \mathscr{J})$ be a JAP. A *judgment aggregation correspondence* (JAC) is a function $F : \mathscr{J}^N \to 2^{\mathscr{J}}$, assigning a set of judgments to each judgment profile.

Definition 12. *Choice by plurality voting (CPV)* is the aggregation correspondence F defined by

$$F(J^N) = \{J^i, i \in N : J^i \in J^N \text{ and } \mid j' : J^{j'} = J^j \mid \leq \mid j' : J^{j'} = J^i \mid,\ \forall j \in N\}$$

In words, given a judgment profile, the AC chooses those judgments in the profile that are shared by the largest number of judges. This aggregation correspondence shares the following properties:

- *Anonymity*: For all profiles $J^N \in \mathscr{J}^N$ and for all permutations π of $N = \{1, 2, \ldots, n\}$,

  $F(J^{\pi(1)}, \ldots, J^{\pi(n)}) = F(J^1, \ldots, J^n)$.
- *Neutrality*: For all permutations σ of $\mathscr{J}$ and for all profiles $J^N \in \mathscr{J}^N$,

  $F(\sigma(J^1), \ldots, \sigma(J^n)) = \sigma(F(J^1, \ldots, J^n))$.
- *Unanimity*: For all judgments $J \in \mathscr{J}$,

  $F(J, \ldots, J) = \{J\}$.
- *Reinforcement*: Let $g_1 = (N, A_k, \neg, \wedge, \mathscr{J})$ and $g_2 = (M, A_k, \neg, \wedge, \mathscr{J})$ be two judgment aggregation problems with the same agenda and disjoint sets of judges, N and M; $N \cap M = \emptyset$.
  If $F(J^N) \cap F(J^M) \neq \emptyset$, then (in JAP $g_3 = (N \cup M, A_k, \neg, \wedge, \mathscr{J})$),

  $F(J^N, J^M) = F(J^N) \cap F(J^M)$.

Theorem 3. *The choice by plurality voting is the only judgment aggregation correspondence that satisfies anonymity, neutrality, unanimity, and reinforcement.*

Proof. This follows readily from Roberts (1991) who, following Young (1975) and Richelson (1978), considered a choice function (or correspondence) from an abstract set X of alternatives and any number of voters: $f : \bigcup_{n=1}^{\infty} X^n \rightarrow P_0(X)$, where $P_0(X)$ is the set of nonempty subsets of X. Roberts provided several sets of axioms characterizing the CPV correspondence in his abstract aggregated choice model. Our characterization theorem is a special case of Roberts's results for $X = \mathscr{J}$ that states that our stated properties, *anonymity, neutrality, unanimity*, and *reinforcement*, characterize the CPV correspondence (Theorem 3 (case 4) in Roberts, 1991). ■

Example 9 (*The Doctrinal Paradox Revisited*). For the classical example of the Doctrinal Paradox,

	Issues					
	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	0	1	1	0	0	1

we have $F(pqg, p\neg q\neg g, \neg pq\neg g) = \{pqg, p\neg q\neg g, \neg pq\neg g\}$.

In other words, the judgment of each of the judges can be chosen.

Consider now the following variant of the situation with five judges:

	Issues					
	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	1	0	1	0
Judge 3	1	0	0	1	0	1
Judge 4	0	1	1	0	0	1
Judge 5	0	1	0	1	0	1

We note that the same "paradox" persists, but now $F(J^N) = \{pqg\}$. In particular, the verdict is *Guilty*.

Consider now the following variant of the situation with five judges:

	Issues					
	p	$\neg p$	q	$\neg q$	g	$\neg g$
Judge 1	1	0	1	0	1	0
Judge 2	1	0	0	1	0	1
Judge 3	1	0	0	1	0	1
Judge 4	0	1	1	0	0	1
Judge 5	0	1	1	0	0	1

Again, the same "paradox" persists, but now $F(J^N) = \{p\neg q\neg g, \neg pq\neg g\}$. In particular, the verdict is *'Not guilty'*.

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Appendix

Proof of Claim 1. The issue I_{10} is not relevant to the issue I_{44}, that is, $I_{10} \notin R(I_{44})$.

Proof. The proof is by straightforward verification noticing that $\neg q_{10} = q_1^c$, $\neg q_{01} = q_1'^c$, $\neg q_{kk} = q_k^c \cup q_k'^c$, and using the entailments established in Example 7.

- $q_{10} \wedge q_{01} = q_1 \cup q_1' \not\models I_{44}$.
- $q_{10} \wedge \neg q_{01} = \neg q_1' \not\models I_{44}$.
- $\neg q_{10} \wedge q_{01} = \neg q_1 \not\models I_{44}$.
- $\neg q_{10} \wedge \neg q_{01} = \emptyset \not\models I_{44}$.

- $q_{10} \wedge q_{22} = q_2' \not\models I_{44}$.
- $q_{10} \wedge \neg q_{22} = q_1 \cup \neg q_2' \not\models I_{44}$.
- $\neg q_{10} \wedge q_{22} = q_2 \not\models I_{44}$.
- $\neg q_{10} \wedge \neg q_{22} = \{f, g, h, m\} \not\models I_{44}$.

- $q_{10} \wedge q_{33} = q_1 \cup q_3' \not\models I_{44}$.
- $q_{10} \wedge \neg q_{33} = \neg q_3' \not\models I_{44}$.
- $\neg q_{10} \wedge q_{33} = \{c, f, g\} \not\models I_{44}$.
- $\neg q_{10} \wedge \neg q_{33} = \neg q_3 \not\models I_{44}$.

We proceed to check the implications of the triples of issues involving I_{10}.

- Propositions from I_{10}, I_{01}, I_{22}.
  - $q_{10} \wedge q_{01} \wedge q_{22} = \emptyset \not\models I_{44}$.
  - $q_{10} \wedge q_{01} \wedge \neg q_{22} = q_1 \cup q_1' \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge q_{22} = q_2' \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge \neg q_{22} = \{f', g', h', m'\} \not\models I_{44}$.

  - $\neg q_{10} \wedge q_{01} \wedge q_{22} = q_2 \not\models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge \neg q_{22} = \{f, g, h, m\} \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{01} \wedge q_{22} = \emptyset \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{01} \wedge \neg q_{22} = \emptyset \not\models I_{44}$.
- Propositions from I_{10}, I_{01}, I_{33}.
  - $q_{10} \wedge q_{01} \wedge q_{33} = q_1 \cup q_1' \not\models I_{44}$.
  - $q_{10} \wedge q_{01} \wedge \neg q_{33} = \emptyset \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge q_{33} = \{c', f', g'\} \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge \neg q_{33} = \neg q_3'^c \not\models I_{44}$.

  - $\neg q_{10} \wedge q_{01} \wedge q_{33} = \{c, f, g\} \not\models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge \neg q_{33} = \neg q_3^c \not\models I_{44}$.

  - $\neg q_{10} \wedge \neg q_{01} \wedge q_{33} = \emptyset \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{01} \wedge \neg q_{33} = \emptyset \not\models I_{44}$.
- Propositions from I_{10}, I_{22}, I_{33}.
  - $q_{10} \wedge q_{22} \wedge q_{33} = \{c'\} \models I_{44}$,
  but this does not imply the relevance of q_{10} to q_{44} since $q_{22} \wedge q_{33} = \{c, c'\} \models I_{44}$.
  - $q_{10} \wedge q_{22} \wedge \neg q_{33} = \{d', e'\} \models I_{44}$,
  but this does not imply the relevance of q_{10} to q_{44} since $q_{22} \wedge \neg q_{33} = \{d, d', e, e'\} \models I_{44}$.
  - $q_{10} \wedge \neg q_{22} \wedge q_{33} = \{a, b, a', b', f', g'\} \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{22} \wedge \neg q_{33} = \{h', m'\} \not\models I_{44}$.

  - $\neg q_{10} \wedge q_{22} \wedge q_{33} = \{c\} \models I_{44}$,
  but this does not imply the relevance of q_{10} to q_{44} since $q_{22} \wedge q_{33} = \{c, c'\} \models I_{44}$.
  - $\neg q_{10} \wedge q_{22} \wedge \neg q_{33} = \{d, e\} \models I_{44}$,
  but this does not imply the relevance of q_{10} to q_{44} since $q_{22} \wedge \neg q_{33} = \{d, d', e, e'\} \models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{22} \wedge q_{33} = \{f, g\} \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{22} \wedge \neg q_{33} = \{h, m\} \not\models I_{44}$.

Finally, we check the implications of the quadruples of issues involving I_{10}.

  - $q_{10} \wedge q_{01} \wedge q_{22} \wedge q_{33} = \emptyset \not\models I_{44}$.
  - $q_{10} \wedge q_{01} \wedge q_{22} \wedge \neg q_{33} = \emptyset \not\models I_{44}$.
  - $q_{10} \wedge q_{01} \wedge \neg q_{22} \wedge q_{33} = \{a, b, a', b'\} \not\models I_{44}$.
  - $q_{10} \wedge q_{01} \wedge \neg q_{22} \wedge \neg q_{33} = \emptyset \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge q_{22} \wedge q_{33} = \{c'\} \models I_{44}$,
  but this does not imply the relevance of q_{10} to q_{44} since $\neg q_{01} \wedge q_{22} \wedge q_{33} = \{c'\} \models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge q_{22} \wedge \neg q_{33} = \{d', e'\} \models I_{44}$,
  but this does not imply the relevance of q_{10} to q_{44} since $\neg q_{01} \wedge q_{22} \wedge \neg q_{33} = \{d', e'\} \models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge \neg q_{22} \wedge q_{33} = \{f', g'\} \not\models I_{44}$.
  - $q_{10} \wedge \neg q_{01} \wedge \neg q_{22} \wedge \neg q_{33} = \{h', m'\} \not\models I_{44}$.

  - $\neg q_{10} \wedge q_{01} \wedge q_{22} \wedge q_{33} = \{c\} \models I_{44}$,
  but this does not imply the relevance of q_{10} to q_{44} since $q_{01} \wedge q_{22} \wedge q_{33} = \{c\} \models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge q_{22} \wedge \neg q_{33} = \{d, e\} \models I_{44}$,
  but this does not imply the relevance of q_{10} to q_{44} since $q_{01} \wedge q_{22} \wedge \neg q_{33} = \{d, e\} \models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge \neg q_{22} \wedge q_{33} = \{f, g\} \not\models I_{44}$.
  - $\neg q_{10} \wedge q_{01} \wedge \neg q_{22} \wedge \neg q_{33} = \{h, m\} \not\models I_{44}$.
  - $\neg q_{10} \wedge \neg q_{01} = \emptyset$, eliminating the remaining four cases $\neg q_{10} \wedge \neg q_{01} \wedge \{\pm q_{22}\} \wedge \{\pm q_{33}\}$. ■

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